

On the Capacity of the K -User Cyclic Gaussian Interference Channel

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Abstract—This paper studies the capacity region of a K -user cyclic Gaussian interference channel, where the k th user interferes with only the $(k-1)$ th user ($\text{mod } K$) in the network. Inspired by the work of Etkin, Tse and Wang, who derived a capacity region outer bound for the two-user Gaussian interference channel and proved that a simple Han-Kobayashi power splitting scheme can achieve to within one bit of the capacity region for all values of channel parameters, this paper shows that a similar strategy also achieves the capacity region of the K -user cyclic interference channel to within a constant gap in the weak interference regime. Specifically, for the K -user cyclic Gaussian interference channel, a compact representation of the Han-Kobayashi achievable rate region using Fourier-Motzkin elimination is first derived, a capacity region outer bound is then established. It is shown that the Etkin-Tse-Wang power splitting strategy gives a constant gap of at most 2 bits in the weak interference regime. For the special 3-user case, this gap can be sharpened to $1\frac{1}{2}$ bits by time-sharing of several different strategies. The capacity result of the K -user cyclic Gaussian interference channel in the strong interference regime is also given. Further, based on the capacity results, this paper studies the generalized degrees of freedom (GDoF) of the symmetric cyclic interference channel. It is shown that the GDoF of the symmetric capacity is the same as that of the classic two-user interference channel, no matter how many users are in the network.

Index Terms—Approximate capacity, Han-Kobayashi, Fourier-Motzkin, K -user interference channel, multicell processing.

I. INTRODUCTION

The interference channel models a communication scenario in which several mutually interfering transmitter-receiver pairs share the same physical medium. The interference channel is a useful model for many practical systems such as the wireless network. The capacity region of the interference channel, however, has not been completely characterized, even for the two-user Gaussian case.

The largest known achievable rate region for the two-user interference channel is given by Han and Kobayashi [1] using a coding scheme involving common-private power splitting. Chong et al. [2] obtained the same rate region in a simpler form by applying the Fourier-Motzkin algorithm together with

Manuscript received October 4, 2010; revised May 8, 2012; accepted August 28, 2012. Date of current version August 30, 2012. This work was supported by the Natural Science and Engineering Research Council (NSERC). The material in this paper was presented in part at the 2010 IEEE Conference on Information Science and Systems (CISS), and in part at the 2011 IEEE Symposium of Information Theory (ISIT).

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a time-sharing technique to the Han and Kobayashi's rate region characterization. The optimality of the Han-Kobayashi region for the two-user Gaussian interference channel is still an open problem in general, except in the strong interference regime where transmission with common information only achieves the capacity region [1], [3], [4], and in a noisy interference regime where transmission with private information only achieves the sum capacity [5]–[7].

In a breakthrough, Etkin, Tse and Wang [8] showed that the Han-Kobayashi scheme can in fact achieve to within one bit of the capacity region for the two-user Gaussian interference channel for all channel parameters. Their key insight was that the interference-to-noise ratio (INR) of the private message should be chosen to be as close to 1 as possible in the Han-Kobayashi scheme. They also found a new capacity region outer bound using a genie-aided technique. In the rest of this paper, we refer this particular setting of the private message power as the Etkin-Tse-Wang (ETW) power-splitting strategy.

The Etkin, Tse and Wang's result applies only to the two-user interference channel. Practical systems often have more than two transmitter-receiver pairs, yet the generalization of Etkin, Tse and Wang's work to the interference channels with more than two users has proved difficult for the following reasons. First, it appears that the Han-Kobayashi common-private superposition coding is no longer adequate for the K -user interference channel. Interference alignment types of coding scheme [9] [10] can potentially enlarge the achievable rate region. Second, even within the Han-Kobayashi framework, when more than two receivers are involved, multiple common messages at each transmitter may be needed, making the optimization of the resulting rate region difficult.

In the context of K -user Gaussian interference channels, sum capacity results are available in the noisy interference regime [5], [11]. In particular, Annapureddy et al. [5] obtained the sum capacity for the symmetric three-user Gaussian interference channel, the one-to-many, and the many-to-one Gaussian interference channels under the noisy interference criterion. Similarly, Shang et al. [11] studied the fully connected K -user Gaussian interference channel and showed that treating interference as noise at the receiver is sum-capacity achieving when the transmit power and the cross channel gains are sufficiently weak to satisfy a certain criterion. Further, achievability and outer bounds for the three-user interference channel have also been studied in [12] and [13]. Finally, much work has been carried out on the generalized degree of freedom (GDoF as defined in [8]) of the K -user interference channel and its variations [9], [14]–[16].

Instead of treating the general K -user interference channel, this paper focuses on a cyclic Gaussian interference channel,

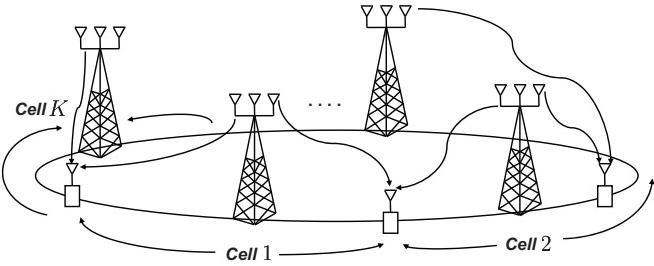


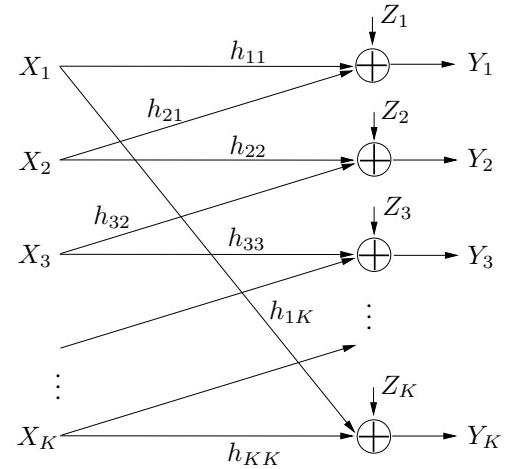
Fig. 1. The circular array soft-handoff model

where the k th user interferes with only the $(k - 1)$ th user. In this case, each transmitter interferes with only one other receiver, and each receiver suffers interference from only one other transmitter, thereby avoiding the difficulties mentioned earlier. For the K -user cyclic interference channel, the Etkin, Tse and Wang's coding strategy remains a natural one. The main objective of this paper is to show that it indeed achieves to within a constant gap of the capacity region for this cyclic model in the weak interference regime to be defined later.

The cyclic interference channel model is motivated by the so-called modified Wyner model, which describes the soft handoff scenario of a cellular network [17]. The original Wyner model [18] assumes that all cells are arranged in a linear array with the base-stations located at the center of each cell, and where intercell interference comes from only the two adjacent cells. In the modified Wyner model [17] cells are arranged in a circular array as shown in Fig. 1. The mobile terminals are located along the circular array. If one assumes that the mobile terminals always communicate with the intended base-station to their left (or right), while only suffering from interference due to the base-station to their right (or left), one arrives at the K -user cyclic Gaussian interference channel studied in this paper. The modified Wyner model has been extensively studied in the literature [17], [19], [20], but often either with interference treated as noise or with the assumption of full base-station cooperation. This paper studies the modified Wyner model without base-station cooperation, in which case the soft-handoff problem becomes that of a cyclic interference channel.

This paper primarily focuses on the K -user cyclic Gaussian interference channel in the weak interference regime. The main contributions of this paper are as follows. This paper first derives a compact characterization of the Han-Kobayashi achievable rate region by applying the Fourier-Motzkin elimination algorithm. A capacity region outer bound is then obtained. It is shown that with the Etkin, Tse and Wang's coding strategy, one can achieve to within $1\frac{1}{2}$ bits of the capacity region when $K = 3$ (with time-sharing), and to within two bits of the capacity region in general in the weak interference regime. Finally, the capacity result for the strong interference regime is also derived.

A key part of the development involves a Fourier-Motzkin elimination procedure on the achievable rate region of the K -user cyclic interference channel. To deal with the large number of inequality constraints, an induction proof is used. It is shown that as compared to the two-user case, where the

Fig. 2. K -user cyclic Gaussian interference channel

rate region is defined by constraints on the individual rate R_i , the sum rate $R_1 + R_2$, and the sum rate plus an individual rate $2R_i + R_j$ ($i \neq j$), the achievable rate region for the K -user cyclic interference channel is defined by an additional set of constraints on the sum rate of any arbitrary l adjacent users, where $2 \leq l < K$. These four types of rate constraints completely characterize the Han-Kobayashi region for the K -user cyclic interference channel. They give rise to a total of $K^2 + 1$ constraints.

For the symmetric K -user cyclic channel where all direct links share the same channel gain and all cross links share another channel gain, it is shown that the GDoF of the symmetric capacity is not dependent on the number of users in the network. Therefore, adding more users to a K -user cyclic interference channel with symmetric channel parameters does not affect the per-user rate.

II. CHANNEL MODEL

The K -user cyclic Gaussian interference channel (as depicted in Fig. 2) has K transmitter-receiver pairs. Each transmitter tries to communicate with its intended receiver while causing interference to only one neighboring receiver. Each receiver receives a signal intended for it and an interference signal from only one neighboring sender plus an additive white Gaussian noise (AWGN). As shown in Fig. 2, X_1, X_2, \dots, X_K and Y_1, Y_2, \dots, Y_K are the complex-valued input and output signals, respectively, and $Z_i \sim \mathcal{CN}(0, \sigma^2)$ is the independent and identically distributed (i.i.d) Gaussian noise at receiver i . The input-output model can be written as

$$\begin{aligned} Y_1 &= h_{1,1}X_1 + h_{2,1}X_2 + Z_1, \\ Y_2 &= h_{2,2}X_2 + h_{3,2}X_3 + Z_2, \\ &\vdots \\ Y_K &= h_{K,K}X_K + h_{1,K}X_1 + Z_K, \end{aligned}$$

where each X_i has a power constraint P_i associated with it, i.e., $\mathbb{E}[|x_i|^2] \leq P_i$. Here, $h_{i,j}$ is the channel gain from transmitter i to receiver j .

Define the signal-to-noise and interference-to-noise ratios for each user as follows:

$$\text{SNR}_i = \frac{|h_{i,i}|^2 P_i}{\sigma^2} \quad \text{INR}_i = \frac{|h_{i,i-1}|^2 P_i}{\sigma^2}, \quad i = 1, 2, \dots, K. \quad (1)$$

The K -user cyclic Gaussian interference channel is said to be in the *weak* interference regime if

$$\text{INR}_i \leq \text{SNR}_i, \quad \forall i = 1, 2, \dots, K. \quad (2)$$

and the *strong* interference regime if

$$\text{INR}_i \geq \text{SNR}_i, \quad \forall i = 1, 2, \dots, K. \quad (3)$$

Otherwise, it is said to be in the *mixed* interference regime. Throughout this paper, modulo arithmetic is implicitly used on the user indices, e.g., $K+1=1$ and $1-1=K$. Note that when $K=2$, the cyclic channel reduces to the conventional two-user interference channel.

III. WITHIN TWO BITS OF THE CAPACITY REGION IN THE WEAK INTERFERENCE REGIME

The generalization of Etkin, Tse and Wang's result to the capacity region of a general (nonsymmetric) K -user cyclic Gaussian interference channel is significantly more complicated. In the two-user case, the shape of the Han-Kobayashi achievable rate region is the union of polyhedrons (each corresponding to a fixed input distribution) with boundaries defined by rate constraints on $R_1, R_2, R_1+R_2, 2R_1+R_2$ and $2R_2+R_1$, respectively. In the multiuser case, to extend Etkin, Tse and Wang's result, one needs to find a similar rate region characterization for the general K -user cyclic interference channel first.

A key feature of the cyclic Gaussian interference channel model is that each transmitter sends signal to its intended receiver while causing interference to *only one* of its neighboring receivers; meanwhile, each receiver receives the intended signal plus the interfering signal from *only one* of its neighboring transmitters. Using this fact and with the help of Fourier-Motzkin elimination algorithm, this section shows that the achievable rate region of the K -user cyclic Gaussian interference channel is the union of polyhedrons with boundaries defined by rate constraints on the individual rates R_i , the sum rate R_{sum} , the sum rate plus an individual rate $R_{sum} + R_i$ ($i = 1, 2, \dots, K$), and the sum rate for arbitrary l adjacent users ($2 \leq l < K$). This last rate constraint on arbitrary l adjacent users' rates is new as compared with the two-user case.

The preceding characterization together with outer bounds to be proved later in the section allows us to prove that the capacity region of the K -user cyclic Gaussian interference channel can be achieved to within a constant gap using the ETW power-splitting strategy in the weak interference regime. However, instead of the one-bit result for the two-user interference channel, this section shows that one can achieve to within $1\frac{1}{2}$ bits of the capacity region when $K=3$ (with time-sharing), and within two bits of the capacity region for general K . Again, the strong interference regime is treated later.

A. Achievable Rate Region

Theorem 1: Let \mathcal{P} denote the set of probability distributions $P(\cdot)$ that factor as

$$\begin{aligned} P(q, w_1, x_1, w_2, x_2, \dots, w_K, x_K) \\ = p(q)p(x_1 w_1 | q)p(x_2 w_2 | q) \cdots p(x_K w_K | q). \end{aligned} \quad (4)$$

For a fixed $P \in \mathcal{P}$, let $\mathcal{R}_{HK}^{(K)}(P)$ be the set of all rate tuples (R_1, R_2, \dots, R_K) satisfying

$$0 \leq R_i \leq \min\{d_i, a_i + e_{i-1}\}, \quad (5)$$

$$\sum_{j=m}^{m+l-1} R_j \leq \min \left\{ g_m + \sum_{j=m+1}^{m+l-2} e_j + a_{m+l-1}, \right. \\ \left. \sum_{j=m-1}^{m+l-2} e_j + a_{m+l-1} \right\}, \quad (6)$$

$$\sum_{j=1}^K R_j \leq \min \left\{ \sum_{j=1}^K e_j, r_1, r_2, \dots, r_K \right\}, \quad (7)$$

$$\sum_{j=1}^K R_j + R_i \leq a_i + g_i + \sum_{j=1, j \neq i}^K e_j, \quad (8)$$

where a_i, d_i, e_i, g_i and r_i are defined as follows:

$$a_i = I(Y_i; X_i | W_i, W_{i+1}, Q) \quad (9)$$

$$d_i = I(Y_i; X_i | W_{i+1}, Q) \quad (10)$$

$$e_i = I(Y_i; W_{i+1}, X_i | W_i, Q) \quad (11)$$

$$g_i = I(Y_i; W_{i+1}, X_i | Q) \quad (12)$$

$$r_i = a_{i-1} + g_i + \sum_{j=1, j \notin \{i, i-1\}}^K e_j, \quad (13)$$

and the range of indices are $i, m = 1, 2, \dots, K$ in (5) and (8), $l = 2, 3, \dots, K-1$ in (6). Define

$$\mathcal{R}_{HK}^{(K)} = \bigcup_{P \in \mathcal{P}} \mathcal{R}_{HK}^{(K)}(P). \quad (14)$$

Then $\mathcal{R}_{HK}^{(K)}$ is an achievable rate region for the K -user cyclic interference channel¹.

Proof: The achievable rate region can be proved by the Fourier-Motzkin algorithm together with an induction step. The proof follows the Kobayashi and Han's strategy [22] of eliminating a common message at each step. The details are presented in Appendix A. ■

In the above achievable rate region, (5) is the constraint on the achievable rate of an individual user, (6) is the constraint on the achievable sum rate for any l adjacent users ($2 \leq l < K$), (7) is the constraint on the achievable sum rate of all K users, and (8) is the constraint on the achievable sum rate for all K users plus a repeated one. We can also think of (5)-(8) as the sum-rate constraints for arbitrary l adjacent users, where $l=1$ for (5), $2 \leq l < K$ for (6), $l=K$ for (7) and $l=K+1$ for (8).

From (5) to (8), there are a total of $K+K(K-2)+1+K=K^2+1$ constraints. Together they describe the shape of the

¹The same achievable rate region has been found independently in [21].

achievable rate region under a fixed input distribution. The quadratic growth in the number of constraints as a function of K makes the Fourier-Motzkin elimination of the Han-Kobayashi region quite complex. The proof in Appendix A uses induction to deal with the large number of the constraints.

As an example, for the two-user Gaussian interference channel, there are $2^2 + 1 = 5$ rate constraints, corresponding to that of $R_1, R_2, R_1 + R_2, 2R_1 + R_2$ and $2R_2 + R_1$, as in [1], [2], [8], [22]. Specifically, substituting $K = 2$ in Theorem 1 gives us the following achievable rate region:

$$0 \leq R_1 \leq \min\{d_1, a_1 + e_2\}, \quad (15)$$

$$0 \leq R_2 \leq \min\{d_2, a_2 + e_1\}, \quad (16)$$

$$R_1 + R_2 \leq \min\{e_1 + e_2, a_1 + g_2, a_2 + g_1\}, \quad (17)$$

$$2R_1 + R_2 \leq a_1 + g_1 + e_2, \quad (18)$$

$$2R_2 + R_1 \leq a_2 + g_2 + e_1. \quad (19)$$

The above region for the two-user Gaussian interference channel is exactly that of Theorem D in [22].

B. Capacity Region Outer Bound

Theorem 2: For the K -user cyclic Gaussian interference channel in the weak interference regime, the capacity region is included in the set of rate tuples (R_1, R_2, \dots, R_K) such that

$$R_i \leq \lambda_i, \quad (20)$$

$$\sum_{j=m}^{m+l-1} R_j \leq \min \left\{ \gamma_m + \sum_{j=m+1}^{m+l-2} \alpha_j + \beta_{m+l-1}, \mu_m + \sum_{j=m}^{m+l-2} \alpha_j + \beta_{m+l-1} \right\}, \quad (21)$$

$$\sum_{j=1}^K R_j \leq \min \left\{ \sum_{j=1}^K \alpha_j, \rho_1, \rho_2, \dots, \rho_K \right\}, \quad (22)$$

$$\sum_{j=1}^K R_j + R_i \leq \beta_i + \gamma_i + \sum_{j=1, j \neq i}^K \alpha_j, \quad (23)$$

where the ranges of the indices i, m, l are as defined in Theorem 1, and

$$\alpha_i = \log \left(1 + \text{INR}_{i+1} + \frac{\text{SNR}_i}{1 + \text{INR}_i} \right) \quad (24)$$

$$\beta_i = \log \left(\frac{1 + \text{SNR}_i}{1 + \text{INR}_i} \right) \quad (25)$$

$$\gamma_i = \log (1 + \text{INR}_{i+1} + \text{SNR}_i) \quad (26)$$

$$\lambda_i = \log(1 + \text{SNR}_i) \quad (27)$$

$$\mu_i = \log(1 + \text{INR}_i) \quad (28)$$

$$\rho_i = \beta_{i-1} + \gamma_i + \sum_{j=1, j \notin \{i, i-1\}}^K \alpha_j. \quad (29)$$

Proof: See Appendix B. ■

C. Capacity Region to Within Two Bits

Theorem 3: For the K -user cyclic Gaussian interference channel in the weak interference regime, the fixed ETW power-splitting strategy achieves to within two bits of the capacity region².

Proof: Applying the ETW power-splitting strategy (i.e., $\text{INR}_{ip} = \min(\text{INR}_i, 1)$) to Theorem 1, parameters a_i, d_i, e_i, g_i can be easily calculated as follows:

$$a_i = \log(2 + \text{SNR}_{ip}) - 1, \quad (30)$$

$$d_i = \log(2 + \text{SNR}_i) - 1, \quad (31)$$

$$e_i = \log(1 + \text{INR}_{i+1} + \text{SNR}_{ip}) - 1, \quad (32)$$

$$g_i = \log(1 + \text{INR}_{i+1} + \text{SNR}_i) - 1, \quad (33)$$

where $\text{SNR}_{ip} = |h_{i,i}|^2 P_{ip} / \sigma^2$. To prove that the achievable rate region in Theorem 1 with the above a_i, d_i, e_i, g_i is within two bits of the outer bound in Theorem 2, we show that each of the rate constraints in (5)-(8) is within two bits of their corresponding outer bound in (20)-(23) in the weak interference regime, i.e., the following inequalities hold for all i, m, l in the ranges defined in Theorem 1:

$$\delta_{R_i} \leq 2, \quad (34)$$

$$\delta_{R_m + \dots + R_{m+l-1}} \leq 2l, \quad (35)$$

$$\delta_{R_{sum}} \leq 2K, \quad (36)$$

$$\delta_{R_{sum} + R_i} \leq 2(K+1), \quad (37)$$

where $\delta_{(.)}$ is the difference between the achievable rate in Theorem 1 and its corresponding outer bound in Theorem 2. The proof makes use of a set of inequalities provided in Appendix D.

For δ_{R_i} , we have

$$\begin{aligned} \delta_{R_i} &= \lambda_i - \min\{d_i, a_i + e_{i-1}\} \\ &= \max\{\lambda_i - d_i, \lambda_i - (a_i + e_{i-1})\} \\ &\leq 2. \end{aligned} \quad (38)$$

For $\delta_{R_m + \dots + R_{m+l-1}}$, compare the first terms of (6) and (21):

$$\begin{aligned} \delta_1 &= \gamma_m + \sum_{j=m+1}^{m+l-2} \alpha_j + \beta_{m+l-1} - g_m + \sum_{j=m+1}^{m+l-2} e_j \\ &\quad + a_{m+l-1} \\ &= (\gamma_m - g_m) + \sum_{j=m+1}^{m+l-2} (\alpha_j - e_j) + (\beta_{m+l-1} - a_{m+l-1}) \\ &\leq l. \end{aligned} \quad (39)$$

Similarly, the difference between the second term of (6) and

²This paper follows the definition from [8] that if a rate tuple (R_1, R_2, \dots, R_K) is achievable and $(R_1 + b, R_2 + b, \dots, R_K + b)$ is outside the capacity region, then (R_1, R_2, \dots, R_K) is within b bits of the capacity region.

(21) is bounded by

$$\begin{aligned} \delta_2 &= \mu_m + \sum_{j=m}^{m+l-2} \alpha_j + \beta_{m+l-1} - \sum_{j=m-1}^{m+l-2} e_j + a_{m+l-1} \\ &= (\mu_m - e_{m-1}) + \sum_{j=m}^{m+l-2} (\alpha_j - e_j) \\ &\quad + (\beta_{m+l-1} - a_{m+l-1}) \\ &\leq l + 1. \end{aligned} \quad (40)$$

Finally, applying the fact that

$$\min\{x_1, y_1\} - \min\{x_2, y_2\} \leq \max\{x_1 - x_2, y_1 - y_2\},$$

we obtain

$$\delta_{R_m+\dots+R_{m+l-1}} \leq \max\{\delta_1, \delta_2\} \leq l + 1. \quad (41)$$

For $\delta_{R_{sum}}$, the difference between the first terms of (7) and (22) is bounded by

$$\sum_{j=1}^K \alpha_j - \sum_{j=1}^K e_j = \sum_{j=1}^K (\alpha_j - e_j) \leq K. \quad (42)$$

In addition,

$$\begin{aligned} \rho_i - r_i &= \beta_{i-1} + \gamma_i + \sum_{j=1, j \notin \{i, i-1\}}^K \alpha_j \\ &\quad - a_{i-1} + g_i + \sum_{j=1, j \notin \{i, i-1\}}^K e_j \\ &= (\beta_{i-1} - a_{i-1}) + (\gamma_i - g_i) \\ &\quad + \sum_{j=1, j \notin \{i, i-1\}}^K (\alpha_j - e_j) \\ &\leq K \end{aligned} \quad (43)$$

for $i = 1, 2, \dots, K$. As a result, the gap on the sum rate is bounded by

$$\begin{aligned} \delta_{R_{sum}} &= \min \left\{ \sum_{j=1}^K \alpha_j, \rho_1, \rho_2, \dots, \rho_K \right\} \\ &\quad - \min \left\{ \sum_{j=1}^K e_j, r_1, r_2, \dots, r_K \right\} \\ &\leq \max \left\{ \sum_{j=1}^K (\alpha_j - e_j), \rho_1 - r_1, \right. \\ &\quad \left. \rho_2 - r_2, \dots, \rho_K - r_K \right\} \\ &\leq K. \end{aligned} \quad (44)$$

For $R_{sum} + R_i$, we have

$$\begin{aligned} \delta_{R_{sum}+R_i} &= \beta_i + \gamma_i + \sum_{j=1, j \neq i}^K \alpha_j - a_i + g_i + \sum_{j=1, j \neq i}^K e_j \\ &= (\beta_i - a_i) + (\gamma_i - g_i) + \sum_{j=1, j \neq i}^K (\alpha_j - e_j) \\ &\leq K + 1 \end{aligned} \quad (45)$$

Since the inequalities in (34)-(37) hold for all the ranges of i , m , and l defined in Theorem 1, this proves that the ETW power-splitting strategy achieves to within two bits of the capacity region in the weak interference regime. ■

D. 3-User Cyclic Gaussian Interference Channel Capacity Region to Within $1\frac{1}{2}$ Bits

Chong, Motani and Garg [2] showed that by time-sharing with marginalized versions of the input distribution, the Han-Kobayashi region for the two-user interference channel as stated in (15)-(19) can be further simplified by removing the $a_1 + e_2$ and $a_2 + e_1$ terms from (15) and (16) respectively. The resulting rate region without these two terms is proved to be equivalent to the original Han-Kobayashi region (15)-(19).

This section shows that the aforementioned time-sharing technique can be applied to the 3-user cyclic interference channel (but not to $K \geq 4$). By a similar time-sharing strategy, the second rate constraint on R_1, R_2 and R_3 can be removed, and the achievable rate region can be shown to be within $1\frac{1}{2}$ bits of the capacity region in the weak interference regime.

Theorem 4: Let \mathcal{P}_3 denote the set of probability distributions $P_3(\cdot)$ that factor as

$$\begin{aligned} P_3(q, w_1, x_1, w_2, x_2, w_3, x_3) \\ = p(q)p(x_1w_1|q)p(x_2w_2|q)p(x_3w_3|q). \end{aligned} \quad (46)$$

For a fixed $P_3 \in \mathcal{P}_3$, let $\mathcal{R}_{\text{HK-TS}}^{(3)}(P_3)$ be the set of all rate tuples (R_1, R_2, R_3) satisfying

$$R_i \leq d_i, \quad i = 1, 2, 3, \quad (47)$$

$$R_1 + R_2 \leq \min\{g_1 + a_2, e_3 + e_1 + a_2\}, \quad (48)$$

$$R_2 + R_3 \leq \min\{g_2 + a_3, e_1 + e_2 + a_3\}, \quad (49)$$

$$R_3 + R_1 \leq \min\{g_3 + a_1, e_2 + e_3 + a_1\}, \quad (50)$$

$$\begin{aligned} R_1 + R_2 + R_3 \leq \min\{e_1 + e_2 + e_3, a_3 + g_1 + e_2, \\ a_1 + g_2 + e_3, a_2 + g_3 + e_1\}, \end{aligned} \quad (51)$$

$$2R_1 + R_2 + R_3 \leq a_1 + g_1 + e_2 + e_3, \quad (52)$$

$$R_1 + 2R_2 + R_3 \leq a_2 + g_2 + e_3 + e_1, \quad (53)$$

$$R_1 + R_2 + 2R_3 \leq a_3 + g_3 + e_1 + e_2, \quad (54)$$

where a_i, d_i, e_i, g_i are as defined before. Define

$$\mathcal{R}_{\text{HK-TS}}^{(3)} = \bigcup_{P_3 \in \mathcal{P}_3} \mathcal{R}_{\text{HK-TS}}^{(3)}(P_3). \quad (55)$$

Then, $\mathcal{R}_{\text{HK-TS}}^{(3)}$ is an achievable rate region for the 3-user cyclic Gaussian interference channel. Further, when P_3 is set according to the ETW power-splitting strategy, the rate region $\mathcal{R}_{\text{HK-TS}}^{(3)}(P_3)$ is within $1\frac{1}{2}$ bits of the capacity region in the weak interference regime.

Proof: We prove the achievability of $\mathcal{R}_{\text{HK-TS}}^{(3)}$ by showing that $\mathcal{R}_{\text{HK-TS}}^{(3)}$ is equivalent to $\mathcal{R}_{\text{HK}}^{(3)}$. First, since $\mathcal{R}_{\text{HK}}^{(3)}$ contains an extra constraint on each of R_1, R_2 and R_3 (see (5)), it immediately follows that

$$\mathcal{R}_{\text{HK}}^{(3)} \subseteq \mathcal{R}_{\text{HK-TS}}^{(3)}. \quad (56)$$

In Appendix C, it is shown that the inclusion also holds the other way around. Therefore, $\mathcal{R}_{\text{HK}}^{(3)} = \mathcal{R}_{\text{HK-TS}}^{(3)}$ and as a result, $\mathcal{R}_{\text{HK-TS}}^{(3)}$ is achievable.

Applying the ETW power-splitting strategy (i.e., $\text{INR}_{ip} = \min\{\text{INR}_i, 1\}$ and Q is fixed) to $\mathcal{R}_{\text{HK-TS}}^{(3)}(P_3)$, and following along the same line of the proof of Theorem 3, we obtain

$$\delta_{R_i} \leq 1, \quad (57)$$

$$\delta_{R_i+R_{i+1}} \leq 3, \quad (58)$$

$$\delta_{R_{sum}} \leq 3, \quad (59)$$

$$\delta_{R_{sum}+R_i} \leq 4, \quad (60)$$

where $i = 1, 2, 3$. It then follows that the gap to the capacity region is at most $1\frac{1}{2}$ bits in the weak interference regime. ■

As shown in Appendix C, the rate region (47)-(54) is obtained by taking the union over the achievable rate regions with input distributions P_3, P_3^*, P_3^{**} and P_3^{***} , where P_3^*, P_3^{**} and P_3^{***} are the marginalized versions of P_3 . Thus, to achieve within $1\frac{1}{2}$ bits of the capacity region, one needs to time-share among the ETW power-splitting and its three marginalized variations, rather than using the fixed ETW's input alone.

The key improvement of $\mathcal{R}_{\text{HK-TS}}^{(3)}$ over $\mathcal{R}_{\text{HK}}^{(3)}$ is the removal of term $a_i + e_{i-1}$ in (5) using a time-sharing technique. However, the results in Appendix C hold only for $K = 3$. When $K \geq 4$, it is easy to verify that $\mathcal{R}_{\text{HK-TS}}^{(4)}(P_4)$ is not within the union of $\mathcal{R}_{\text{HK}}^{(4)}(P_4)$ and its marginalized variations, i.e., $\mathcal{R}_{\text{HK}}^{(4)} \not\subseteq \mathcal{R}_{\text{HK-TS}}^{(4)}$. Therefore, the techniques used in this paper only allow the two-bit result to be sharpened to a $1\frac{1}{2}$ -bit result for the three-user cyclic Gaussian interference channel, but not for $K \geq 4$.

IV. CAPACITY REGION IN THE STRONG INTERFERENCE REGIME

The results so far pertain only to the weak interference regime, where $\text{SNR}_i \geq \text{INR}_i, \forall i$. In the strong interference regime, where $\text{SNR}_i \leq \text{INR}_i, \forall i$, the capacity result in [1] [4] for the two-user Gaussian interference channel can be easily extended to the K -user cyclic case.

Theorem 5: For the K -user cyclic Gaussian interference channel in the strong interference regime, the capacity region is given by the set of (R_1, R_2, \dots, R_K) such that³

$$\begin{cases} R_i \leq \log(1 + \text{SNR}_i) \\ R_i + R_{i+1} \leq \log(1 + \text{SNR}_i + \text{INR}_{i+1}), \end{cases} \quad (61)$$

for $i = 1, 2, \dots, K$. In the very strong interference regime where $\text{INR}_i \geq (1 + \text{SNR}_{i-1})\text{SNR}_i, \forall i$, the capacity region is the set of (R_1, R_2, \dots, R_K) with

$$R_i \leq \log(1 + \text{SNR}_i), \quad i = 1, 2, \dots, K. \quad (62)$$

Proof: Achievability: It is easy to see that (61) is in fact the intersection of the capacity regions of K multiple-access channels:

$$\bigcap_{i=1}^K \left\{ (R_i, R_{i+1}) \mid \begin{array}{l} R_i \leq \log(1 + \text{SNR}_i) \\ R_{i+1} \leq \log(1 + \text{INR}_{i+1}) \\ R_i + R_{i+1} \leq \log(1 + \text{SNR}_i + \text{INR}_{i+1}). \end{array} \right\}. \quad (63)$$

Each of these regions corresponds to that of a multiple-access channel with W_i^n and W_{i+1}^n as inputs and Y_i^n as output (with $U_i^n = U_{i+1}^n = \emptyset$). Therefore, the rate region (61) can

be achieved by setting all the input signals to be common messages. This completes the achievability part.

Converse: The converse proof follows the idea of [4]. The key ingredient is to show that for a genie-aided Gaussian interference channel to be defined later, in the strong interference regime, whenever a rate tuple (R_1, R_2, \dots, R_K) is achievable, i.e., X_i^n is decodable at receiver i , X_i^n must also be decodable at $Y_{i-1}^n, i = 1, 2, \dots, K$.

The genie-aided Gaussian interference channel is defined by the Gaussian interference channel (see Fig. 2) with genie X_{i+2}^n given to receiver i . The capacity region of the K -user cyclic Gaussian interference channel must reside inside the capacity region of the genie-aided one.

Assume that a rate tuple (R_1, R_2, \dots, R_K) is achievable for the K -user cyclic Gaussian interference channel. In this case, after X_i^n is decoded, with the knowledge of the genie X_{i+2}^n , receiver i can construct the following signal:

$$\begin{aligned} \tilde{Y}_i^n &= \frac{h_{i+1,i+1}}{h_{i+1,i}}(Y_i^n - h_{i,i}X_i^n) + h_{i+2,i+1}X_{i+2}^n \\ &= h_{i+1,i+1}X_{i+1}^n + h_{i+2,i+1}X_{i+2}^n + \frac{h_{i+1,i+1}}{h_{i+1,i}}Z_i^n, \end{aligned}$$

which contains the signal component of Y_{i+1}^n but with less noise since $|h_{i+1,i}| \geq |h_{i+1,i+1}|$ in the strong interference regime. Now, since X_{i+1}^n is decodable at receiver $i+1$, it must also be decodable at receiver i using the constructed \tilde{Y}_i^n . Therefore, X_i^n and X_{i+1}^n are both decodable at receiver i . As a result, the achievable rate region of (R_i, R_{i+1}) is bounded by the capacity region of the multiple-access channel $(X_i^n, X_{i+1}^n, Y_i^n)$, which is shown in (63). Since (63) reduces to (61) in the strong interference regime, we have shown that (61) is an outer bound of the K -user cyclic Gaussian interference channel in the strong interference regime. This completes the converse proof.

In the very strong interference regime where $\text{INR}_i \geq (1 + \text{SNR}_{i-1})\text{SNR}_i, \forall i$, it is easy to verify that the second constraint in (61) is no longer active. This results in the capacity region (62). ■

V. SYMMETRIC CHANNEL AND GENERALIZED DEGREES OF FREEDOM

Consider the symmetric cyclic Gaussian interference channel, where all the direct links from the transmitters to the receivers share the same channel gain and all the cross links share another same channel gain. In addition, all the input signals have the same power constraint P , i.e., $\mathbb{E}[|X_i|^2] \leq P, \forall i$.

The symmetric capacity of the K -user interference channel is defined as

$$C_{\text{sym}} = \begin{cases} \text{maximize} & \min\{R_1, R_2, \dots, R_K\} \\ \text{subject to} & (R_1, R_2, \dots, R_K) \in \mathcal{R} \end{cases} \quad (64)$$

where \mathcal{R} is the capacity region of the K -user interference channel. For the symmetric interference channel, $C_{\text{sym}} = \frac{1}{K}C_{\text{sum}}$, where C_{sum} is the sum capacity. As a direct consequence of Theorem 3 and Theorem 5, the generalized degree of freedom of the symmetric capacity for the symmetric cyclic channel can be derived as follows.

³This capacity result was also recently obtained in [23].

Corollary 1: For the K -user symmetric cyclic Gaussian interference channel,

$$d_{sym} = \begin{cases} \min \left\{ \max \{\alpha, 1 - \alpha\}, 1 - \frac{\alpha}{2} \right\}, & 0 \leq \alpha < 1 \\ \min \left\{ \frac{\alpha}{2}, 1 \right\}, & \alpha \geq 1 \end{cases} \quad (65)$$

where d_{sym} is the generalized degrees of freedom of the symmetric capacity.

Note that the above d_{sym} for the K -user cyclic interference channel with symmetric channel parameters is the same as that of the two-user interference channel derived in [8].

VI. CONCLUSION

This paper investigates the capacity and the coding strategy for the K -user cyclic Gaussian interference channel. Specifically, this paper shows that in the weak interference regime, the ETW power-splitting strategy achieves to within two bits of the capacity region. Further, in the special case of $K = 3$ and with the help of a time-sharing technique, one can achieve to within $1\frac{1}{2}$ bits of the capacity region in the weak interference regime.

The capacity result for the K -user cyclic Gaussian interference channel in the strong interference regime is a straightforward extension of the corresponding two-user case. However, in the mixed interference regime, although the constant gap result may well continue to hold, the proof becomes considerably more complicated, as different mixed scenarios need to be enumerated and the corresponding outer bounds derived.

APPENDIX

A. Proof of Theorem 1

For the two-user interference channel, Kobayashi and Han [22] gave a detailed Fourier-Motzkin elimination procedure for the achievable rate region. The Fourier-Motzkin elimination for the K -user cyclic interference channel involves K elimination steps. The complexity of the process increases with each step. Instead of manually writing down all the inequalities step by step, this appendix uses mathematical induction to derive the final result.

This achievability proof is based on the application of coding scheme in [2] (also referred as the multi-level coding in [24]) to the multi-user setting. Instead of using the original code construction of [1], the following strategy is used in which each common message $W_i, i = 1, 2, \dots, K$ serves to generate 2^{nT_i} cloud centers $W_i(j), j = 1, 2, \dots, 2^{nT_i}$, each of which is surrounded by 2^{nS_i} codewords $X_i(j, k), k = 1, 2, \dots, 2^{nS_i}$. This results in achievable rate region expressions expressed in terms of (W_i, X_i, Y_i) instead of (U_i, W_i, Y_i) . For the two-user interference channel, Chong, Motani and Garg [2, Lemma 3] made a further simplification to the achievable rate region expression. They observed that in the Han-Kobayashi scheme, the common message W_i is only required to be correctly decoded at the intended receiver Y_i and an incorrectly decoded W_i at receiver Y_{i-1} does not cause an error event. Based on this observation, they concluded that for the multiple-access channel with input (U_i, W_i, W_{i+1}) and output Y_i , the rate constraints on common messages T_i ,

T_{i+1} and $T_i + T_{i+1}$ are in fact irrelevant to the decoding error probabilities and can be removed, i.e., the rates (S_i, T_i, T_{i+1}) are constrained by only the following set of inequalities:

$$S_i \leq a_i = I(Y_i; X_i | W_i, W_{i+1}, Q) \quad (66)$$

$$S_i + T_i \leq d_i = I(Y_i; X_i | W_{i+1}, Q) \quad (67)$$

$$S_i + T_{i+1} \leq e_i = I(Y_i; W_{i+1}, X_i | W_i, Q) \quad (68)$$

$$S_i + T_i + T_{i+1} \leq g_i = I(Y_i; W_{i+1}, X_i | Q) \quad (69)$$

$$S_i, T_i, T_{i+1} \geq 0 \quad (70)$$

Now, compare the K -user cyclic interference channel with the two-user interference channel, it is easy to see that in both channel models, each receiver only sees interference from one neighboring transmitter. This makes the decoding error probability analysis for both channel models the same. Therefore, the set of rates $\mathcal{R}(R_1, R_2, \dots, R_K)$, where $R_i = S_i + T_i$, with (S_i, T_i) satisfy (66)-(70) for $i = 1, 2, \dots, K$, characterizes an achievable rate region for the K -user cyclic interference channel.

The first step of using the Fourier-Motzkin algorithm is to eliminate all private messages S_i by substituting $S_i = R_i - T_i$ into the K polymatroids (66)-(70). This results in the following K polymatroids without S_i :

$$R_i - T_i \leq a_i, \quad (71)$$

$$R_i \leq d_i, \quad (72)$$

$$R_i - T_i + T_{i+1} \leq e_i, \quad (73)$$

$$R_i + T_{i+1} \leq g_i, \quad (74)$$

$$-R_i \leq 0, \quad (75)$$

where $i = 1, 2, \dots, K$.

Next, use Fourier-Motzkin algorithm to eliminate common message rates T_1, T_2, \dots, T_K in a step-by-step process so that after n steps, common variables (T_1, \dots, T_n) are eliminated. The induction hypothesis is the following 5 different groups of inequalities, which is assumed to be obtained at the end of the n th elimination step:

(a) Inequalities not including private or common variables S_i and $T_i, i = 1, 2, \dots, K$:

$$R_i \leq d_i, \quad i = 1, 2, \dots, K \quad (76)$$

$$-R_i \leq 0, \quad i = 1, 2, \dots, n \quad (77)$$

$$R_K + R_1 \leq g_K + a_1, \quad (78)$$

$$R_m \leq a_m + e_{m-1}, \quad (79)$$

$$\sum_{j=l}^m R_j \leq \min \left\{ g_l + \sum_{i=l+1}^{m-1} e_j + a_m, \sum_{j=l-1}^{m-1} e_j + a_m \right\}, \quad (80)$$

$$\sum_{j=1}^m R_j \leq g_1 + \sum_{j=2}^{m-1} e_j + a_m, \quad (81)$$

$$\sum_{j=K}^m R_j \leq g_K + \sum_{j=1}^{m-1} e_j + a_m, \quad (82)$$

where $m = 2, 3, \dots, n$ and $l = 2, 3, \dots, m-1$.

(b) Inequalities including T_K but not including T_{n+1} :

$$R_K - T_K \leq a_K, \quad (83)$$

$$-R_K - T_K \leq 0, \quad (84)$$

$$-T_K \leq 0, \quad (85)$$

$$\sum_{j=K}^p R_j - T_K \leq \sum_{j=K}^{p-1} e_j + a_p, \quad (86)$$

where $p = 1, 2, \dots, n$.

(c) All other inequalities not including T_{n+1} :

$$R_{n+1} + T_{n+2} \leq g_{n+1}, \quad (87)$$

and all the polymatroids in (71)-(75) indexed from $n+2$ to $K-1$.

(d) Inequalities including T_{n+1} with a plus sign:

$$T_{n+1} \leq e_n, \quad (88)$$

$$-R_{n+1} + T_{n+1} \leq 0, \quad (89)$$

$$\sum_{j=l}^n R_j + T_{n+1} \leq \min \left\{ \sum_{j=l-1}^n e_j, g_l + \sum_{j=l+1}^n e_j \right\}, \quad (90)$$

$$\sum_{j=1}^n R_j + T_{n+1} \leq g_1 + \sum_{j=2}^n e_j, \quad (91)$$

$$\sum_{j=K}^n R_j + T_{n+1} \leq g_K + \sum_{j=1}^n e_j, \quad (92)$$

$$\sum_{j=K}^n R_j + T_{n+1} - T_K \leq \sum_{j=K}^n e_j, \quad (93)$$

where l goes from 2 to n .

(e) Inequalities including T_{n+1} with a minus sign:

$$R_{n+1} - T_{n+1} \leq a_{n+1}, \quad (94)$$

$$R_{n+1} - T_{n+1} + T_{n+2} \leq e_{n+1}, \quad (95)$$

$$-T_{n+1} \leq 0. \quad (96)$$

It is easy to verify the correctness of inequalities (76)-(96) for $n=2$. We next show that for $n < K-2$, if at the end of step n , the inequalities in (76)-(96) hold, then they must also hold at the end of step $n+1$. Towards this end, we follow the Fourier-Motzkin algorithm [22] by first adding up all the inequalities in (88)-(93) with each of the inequalities in (94)-(96) to eliminate T_{n+1} . This results in the following three groups of inequalities:

(a) Inequalities due to (94):

$$R_{n+1} \leq a_{n+1} + e_n, \quad (97)$$

$$0 \leq a_{n+1}, \quad (98)$$

$$\sum_{j=l}^{n+1} R_j \leq \min \left\{ \sum_{j=l-1}^n e_j + a_{n+1}, g_l + \sum_{j=l+1}^n e_j + a_{n+1} \right\}, \quad (99)$$

$$\sum_{j=1}^{n+1} R_j \leq g_1 + \sum_{j=2}^n e_j + a_{n+1}, \quad (100)$$

$$\sum_{j=K}^{n+1} R_j \leq g_K + \sum_{j=1}^n e_j + a_{n+1}, \quad (101)$$

$$\sum_{j=K}^{n+1} R_j - T_K \leq \sum_{j=K}^n e_j + a_{n+1}, \quad (102)$$

where $l = 2, 3, \dots, n$.

(b) Inequalities due to (95):

$$R_{n+1} + T_{n+2} \leq e_n + e_{n+1}, \quad (103)$$

$$T_{n+2} \leq e_{n+1}, \quad (104)$$

$$\sum_{j=l}^{n+1} R_j + T_{n+2} \leq \min \left\{ \sum_{j=l-1}^{n+1} e_j, g_l + \sum_{j=l+1}^{n+1} e_j \right\}, \quad (105)$$

$$\sum_{j=1}^{n+1} R_j + T_{n+2} \leq g_1 + \sum_{j=2}^{n+1} e_j, \quad (106)$$

$$\sum_{j=K}^{n+1} R_j + T_{n+2} \leq g_K + \sum_{j=1}^{n+1} e_j, \quad (107)$$

$$\sum_{j=K}^{n+1} R_j + T_{n+2} - T_K \leq \sum_{j=K}^{n+1} e_j, \quad (108)$$

where $l = 2, 3, \dots, n$.

(c) Inequalities due to (96):

$$0 \leq e_n, \quad (109)$$

$$-R_{n+1} \leq 0, \quad (110)$$

$$\sum_{j=l}^n R_j \leq \min \left\{ \sum_{j=l-1}^n e_j, g_l + \sum_{j=l+1}^n e_j \right\} \quad (111)$$

$$\sum_{j=1}^n R_j \leq g_1 + \sum_{j=2}^n e_j, \quad (112)$$

$$\sum_{j=K}^n R_j \leq g_K + \sum_{j=1}^n e_j, \quad (113)$$

$$\sum_{j=K}^n R_j - T_K \leq \sum_{j=K}^n e_j, \quad (114)$$

where $l = 2, 3, \dots, n$.

Inspecting the above three groups of inequalities, we can see that (98) and (109) are obviously redundant. Also, (111)

is redundant due to (80), (112) is redundant due to (81), (113) is redundant due to (82), and (114) is redundant due to (86). Now, with these six redundant inequalities removed, the above three groups of inequalities in (97)-(110) together with (76)-(87) form the set of inequalities at the end of step $n+1$. It can be verified that this new set of inequalities is exactly (76)-(96) with n replaced by $n+1$. This completes the induction part.

Now, we proceed with the $(K-1)$ th step. At the end of this step, T_1, T_2, \dots, T_{K-1} would all be removed and only T_K would remain. Because of the cyclic nature of the channel, the set of inequalities (76)-(96) needs to be modified for this $n = K-1$ case. It can be verified that at the end of the $(K-1)$ th step of Fourier-Motzkin algorithm, we obtain the following set of inequalities:

(a) Inequalities not including T_K : (76)-(82) with n replaced by $K-1$ and

$$\sum_{j=1}^K R_j \leq \sum_{j=1}^K e_j. \quad (115)$$

(b) Inequalities including T_K with a plus sign: (88)-(92) with n replace by $K-1$. Note that, (93) becomes (115) when $n = K-1$.

(c) Inequalities including T_K with a minus sign:

$$R_K - T_K \leq a_K, \quad (116)$$

$$\sum_{j=K}^l R_j - T_K \leq \sum_{j=K}^{l-1} e_j + a_l, \quad (117)$$

$$-T_K \leq 0, \quad (118)$$

where $l = 1, 2, \dots, K-1$.

In the K th step (final step) of the Fourier-Motzkin algorithm, T_K is eliminated by adding each of the inequalities involving T_K with a plus sign and each of the inequalities involving T_K with a minus sign to obtain new inequalities not involving T_K . (This is quite similar to the procedure of obtaining (97)-(114).) Finally, after removing all the redundant inequalities, we obtain the set of inequalities in Theorem 1.

B. Proof of Theorem 2

We will prove the outer bounds from (20) to (23) one by one.

First, (20) is simply the cut-set upper bound for user i .

Second, (21) is the bound on the sum-rate of l adjacent users starting from m . According to Fano's inequality, for a

block of length n , we have

$$\begin{aligned} & n \left(\sum_{j=m}^{m+l-1} R_j - \epsilon_n \right) \\ & \leq \sum_{j=m}^{m+l-1} I(x_j^n; y_j^n) \\ & \stackrel{(a)}{\leq} h(y_m^n) - h(y_m^n | x_m^n) + \sum_{j=m+1}^{m+l-2} I(x_j^n; y_j^n s_j^n) \\ & \quad + I(x_{m+l-1}^n; y_{m+l-1}^n | x_{m+l}^n) \\ & = h(y_m^n) - h(s_{m+1}^n) \\ & \quad + \sum_{j=m+1}^{m+l-2} [h(s_j^n) - h(z_{j-1}^n) + h(y_j^n | s_j^n) - h(s_{j+1}^n)] \\ & \quad + h(h_{m+l-1, m+l-1} x_{m+l-1}^n + z_{m+l-1}^n) - h(z_{m+l-1}^n) \\ & = h(y_m^n) - h(z_{m+l-1}^n) + \sum_{j=m+1}^{m+l-2} [h(y_j^n | s_j^n) - h(z_{j-1}^n)] \\ & \quad + h(h_{m+l-1, m+l-1} x_{m+l-1}^n + z_{m+l-1}^n) \\ & \quad - h(h_{m+l-1, m+l-2} x_{m+l-1}^n + z_{m+l-2}^n) \\ & \stackrel{(b)}{\leq} n \left(\gamma_m + \sum_{j=m+1}^{m+l-2} \alpha_j + \beta_{m+l-1} \right), \end{aligned} \quad (119)$$

where in (a) we give genie s_j^n to y_j^n for $m+1 \leq j \leq m+l-2$ and x_{m+l}^n to y_{m+l-1}^n (genies s_j^n are as defined in [25, Theorem 2]), and (b) comes from the fact [8] that Gaussian inputs maximize 1) entropy $h(y_m^n)$, 2) conditional entropy $h(y_j^n | s_j^n)$ for any j , and 3) entropy difference $h(h_{m+l-1, m+l-1} x_{m+l-1}^n + z_{m+l-1}^n) - h(h_{m+l-1, m+l-2} x_{m+l-1}^n + z_{m+l-2}^n)$. This proves the first bound in (21).

Similarly, the second upper bound of (21) can be obtained by giving genie s_j^n to y_j^n for $m \leq j \leq m+l-2$ and x_{m+l}^n to y_{m+l-1}^n :

$$\begin{aligned} & n \left(\sum_{j=m}^{m+l-1} R_j - \epsilon_n \right) \\ & \leq \sum_{j=m}^{m+l-1} I(x_j^n; y_j^n) \\ & \leq \sum_{j=m}^{m+l-2} I(x_j^n; y_j^n s_j^n) + I(x_{m+l-1}^n; y_{m+l-1}^n | x_{m+l}^n) \\ & = \sum_{j=m}^{m+l-2} [h(s_j^n) - h(z_{j-1}^n) + h(y_j^n | s_j^n) - h(s_{j+1}^n)] \\ & \quad + h(h_{m+l-1, m+l-1} x_{m+l-1}^n + z_{m+l-1}^n) - h(z_{m+l-1}^n) \\ & = h(s_m^n) - h(z_{m+l-1}^n) + \sum_{j=m}^{m+l-2} [h(y_j^n | s_j^n) - h(z_{j-1}^n)] \\ & \quad + h(h_{m+l-1, m+l-1} x_{m+l-1}^n + z_{m+l-1}^n) \\ & \quad - h(h_{m+l-1, m+l-2} x_{m+l-1}^n + z_{m+l-2}^n) \\ & \leq n \left(\mu_m + \sum_{j=m}^{m+l-2} \alpha_j + \beta_{m+l-1} \right). \end{aligned} \quad (120)$$

Combining (119) and (120) gives the upper bound in (21).

Third, the first upper bound in (22) is in fact the non-symmetric version of [25, Theorem 2], from which we have

$$\begin{aligned} R_{sum} - n\epsilon_n &\leq \sum_{k=1}^K \{h(y_{ki}|s_{ki}) - h(z_{ki})\} \\ &\leq n \sum_{j=1}^K \alpha_j. \end{aligned} \quad (121)$$

The other sum-rate upper bounds (i.e., ρ_l) can be derived by giving genies x_l^n to y_{l-1}^n and s_j^n to y_j^n for $j = 1, 2, \dots, K, j \neq l, l-1$:

$$\begin{aligned} &n(R_{sum} - \epsilon_n) \\ &\leq I(x_1^n; y_1^n) + I(x_2^n; y_2^n) + \dots + I(x_K^n; y_K^n) \\ &= I(x_{l-1}^n; y_{l-1}^n | x_l^n) + I(x_l^n; y_l^n) + \sum_{j=1, j \neq l, l-1}^K I(x_j^n; y_j^n s_j^n) \\ &= h(h_{l-1, l-1} x_{l-1}^n + z_{l-1}^n) - h(z_{l-1}^n) + h(y_l^n) - h(s_{l+1}^n) \\ &\quad + \sum_{j=1, j \neq l, l-1}^K [h(s_j^n) - h(z_{j-1}^n) + h(y_j^n | s_j^n) - h(s_{j+1}^n)] \\ &= h(y_l^n) - h(z_{l-1}^n) + h(h_{l-1, l-1} x_{l-1}^n + z_{l-1}^n) \\ &\quad - h(h_{l-1, l-2} x_{l-1}^n + z_{l-2}^n) \\ &\quad + \sum_{j=1, j \neq l, l-1}^K [h(y_j^n | s_j^n) - h(z_{j-1}^n)] \\ &\leq n \left(\beta_{l-1} + \gamma_l + \sum_{j=1, j \neq l, l-1}^K \alpha_j \right) \\ &= n\rho_l \end{aligned} \quad (122)$$

where $l = 1, 2, \dots, K$.

Fourth, for the bound in (23), from Fano's inequality, we have

$$\begin{aligned} &n(R_{sum} + R_i - \epsilon_n) \\ &\leq \sum_{j=1}^K I(x_j^n; y_j^n) + I(x_i^n; y_i^n) \\ &\stackrel{(a)}{\leq} I(x_i^n; y_i^n) + I(x_i^n; y_i^n | x_{i+1}^n) + \sum_{j=1, j \neq i}^K I(x_j^n; y_j^n s_j^n) \\ &= h(y_i^n) - h(s_{i+1}^n) + h(h_{i,i} x_i^n + z_i^n) - h(z_i^n) \\ &\quad + \sum_{j=1, j \neq i}^K [h(s_j^n) - h(z_{j-1}^n) + h(y_j^n | s_j^n) - h(s_{j+1}^n)] \\ &= h(y_i^n) - h(z_i^n) + h(h_{i,i} x_i^n + z_i^n) - h(h_{i,i-1} x_i^n + z_i^n) \\ &\quad + \sum_{j=1, j \neq i}^K [h(y_j^n | s_j^n) - h(z_{j-1}^n)] \\ &\leq n \left(\beta_i + \gamma_i + \sum_{j=1, j \neq i}^K \alpha_j \right) \end{aligned} \quad (123)$$

where in (a) we give genie x_{i+1}^n to y_i^n and s_j^n to y_j^n for $j = 1, 2, \dots, K, j \neq i$.

C. Proof of $\mathcal{R}_{HK-TS}^{(3)} \subseteq \mathcal{R}_{HK}^{(3)}$

For a fixed $P_3 \subseteq \mathcal{P}_3$, define

$$P_3^* = \sum_{w_1} P_3, \quad P_3^{**} = \sum_{w_2} P_3, \quad P_3^{***} = \sum_{w_3} P_3. \quad (124)$$

We will show that

$$\begin{aligned} &\mathcal{R}_{HK-TS}^{(3)}(P_3) \\ &\subseteq \mathcal{R}_{HK}^{(3)}(P_3) \cup \mathcal{R}_{HK}^{(3)}(P_3^*) \cup \mathcal{R}_{HK}^{(3)}(P_3^{**}) \cup \mathcal{R}_{HK}^{(3)}(P_3^{***}). \end{aligned} \quad (125)$$

Suppose that rate triple (R_1, R_2, R_3) is in $\mathcal{R}_{HK-TS}^{(3)}(P_3)$ but not in $\mathcal{R}_{HK}^{(3)}(P_3)$. Then at least one of the following inequalities hold:

$$a_1 + e_3 \leq R_1 \leq d_1, \quad (126)$$

$$a_2 + e_1 \leq R_2 \leq d_2, \quad (127)$$

$$a_3 + e_2 \leq R_3 \leq d_3, \quad (128)$$

Without loss of generality, assume that (126) holds.

Substituting $W_1 = \emptyset$ into $\mathcal{R}_{HK}^{(3)}(P_3)$, we obtain $\mathcal{R}_{HK}^{(3)}(P_3^*)$ as follows:

$$R_1 \leq d_1, \quad (129)$$

$$R_2 \leq \min\{d_2, a_2 + g_1\}, \quad (130)$$

$$\begin{aligned} R_3 &\leq \min\{I(Y_3; X_3|Q), \\ &\quad e_2 + I(Y_3; X_3|W_3, Q)\}, \end{aligned} \quad (131)$$

$$R_1 + R_2 \leq a_2 + g_1, \quad (132)$$

$$\begin{aligned} R_2 + R_3 &\leq \min\{g_2 + I(Y_3; X_3|W_3, Q), \\ &\quad g_1 + e_2 + I(Y_3; X_3|W_3, Q)\}, \end{aligned} \quad (133)$$

$$\begin{aligned} R_3 + R_1 &\leq \min\{d_1 + I(Y_3; X_3|Q), \\ &\quad d_1 + e_2 + I(Y_3; X_3|W_3, Q)\}, \end{aligned} \quad (134)$$

$$R_1 + R_2 + R_3 \leq g_1 + e_2 + I(Y_3; X_3|W_3, Q). \quad (135)$$

We will show that whenever (126) is true, we have $\mathcal{R}_{HK-TS}^{(3)}(P_3) \subseteq \mathcal{R}_{HK}^{(3)}(P_3^*)$. To this end, inspect $\mathcal{R}_{HK-TS}^{(3)}(P_3)$ in (47)-(54). From (47), we have

$$R_1 \leq d_1, \quad (136)$$

and from (47) and (126) and (48), we have

$$\begin{aligned} R_2 &\leq \min\{d_2, a_2 + e_1 - a_1\} \\ &\leq \min\{d_2, a_2 + g_1\}, \end{aligned} \quad (137)$$

and from (126) and (50), we have

$$\begin{aligned} R_3 &\leq \min\{g_3 - e_3, e_2\} \\ &\leq \min\{I(Y_3; X_3|Q), e_2 + I(Y_3; X_3|W_3, Q)\} \end{aligned} \quad (138)$$

and from (48), we have

$$R_1 + R_2 \leq a_2 + g_1, \quad (139)$$

and from (126) and (51), we have

$$\begin{aligned} R_2 + R_3 &\leq \min\{g_2, e_1 + e_2 - a_1\} \\ &\leq \min\{g_2 + I(Y_3; X_3|W_3, Q), \\ &\quad g_1 + e_2 + I(Y_3; X_3|W_3, Q)\}, \end{aligned} \quad (140)$$

and from (126) and (50), we have

$$\begin{aligned} R_3 + R_1 &\leq \min\{d_1 + g_3 - a_3, e_2 + d_1\} \\ &\leq \min\{d_1 + I(Y_3; X_3|Q), \\ &\quad d_1 + e_2 + I(Y_3; X_3|W_3, Q)\}, \end{aligned} \quad (141)$$

and from (126) and (52), we have

$$\begin{aligned} R_1 + R_2 + R_3 &\leq g_1 + e_2 \\ &\leq g_1 + e_2 + I(Y_3; X_3|W_3, Q). \end{aligned} \quad (142)$$

It is easy to see that (R_1, R_2, R_3) satisfying the above constraints (136)-(142) is within the rate region $\mathcal{R}_{HK}^{(3)}(P_3^*)$. In the same way, we can prove the cases for when (127) holds and when (128) holds.

Therefore, (125) is true, and it immediately follows that

$$\mathcal{R}_{HK-TS}^{(3)} \subseteq \mathcal{R}_{HK}^{(3)}. \quad (143)$$

D. Useful Inequalities

Keep in mind that, with the ETW's power splitting strategy, i.e., $\text{SNR}_{ip} = \min\{\text{SNR}_i, \frac{\text{SNR}_i}{1+\text{INR}_i}\}$, we always have $\text{SNR}_{ip} > \frac{\text{SNR}_i}{1+\text{INR}_i}$. This appendix presents several useful inequalities as follows. For all $i = 1, 2, \dots, K$,

- $\lambda_i - d_i < 1$, because

$$\begin{aligned} \lambda_i - d_i &= \log(1 + \text{SNR}_i) - \log(2 + \text{SNR}_i) + 1 \\ &= 1 - \log\left(\frac{2 + \text{SNR}_i}{1 + \text{SNR}_i}\right) \\ &\leq 1 \end{aligned} \quad (144)$$

- $\lambda_i - (a_i + e_{i-1}) < 2$, because

$$\begin{aligned} \lambda_i - (a_i + e_{i-1}) &= \log(1 + \text{SNR}_i) - \log(2 + \text{SNR}_{ip}) + 1 \\ &\quad - \log(1 + \text{INR}_i + \text{SNR}_{i-1,p}) + 1 \\ &< 2 + \log(1 + \text{SNR}_i) - \log\left(1 + \frac{\text{SNR}_i}{1 + \text{INR}_i}\right) \\ &\quad - \log(1 + \text{INR}_i) \\ &= 2 - \log\left(1 + \frac{\text{INR}_i}{1 + \text{SNR}_i}\right) \\ &\leq 2 \end{aligned} \quad (145)$$

- $\beta_i - a_i < 1$, because

$$\begin{aligned} \beta_i - a_i &= \log\left(\frac{1 + \text{SNR}_i}{1 + \text{INR}_i}\right) - \log(2 + \text{SNR}_{ip}) + 1 \\ &< \log\left(\frac{1 + \text{SNR}_i}{1 + \text{INR}_i}\right) - \log\left(1 + \frac{\text{SNR}_i}{1 + \text{INR}_i}\right) + 1 \\ &= 1 - \log\left(1 + \frac{\text{INR}_i}{1 + \text{SNR}_i}\right) \\ &\leq 1 \end{aligned} \quad (146)$$

- $\alpha_i - e_i < 1$, because

$$\begin{aligned} \alpha_i - e_i &= \log\left(1 + \text{INR}_{i+1} + \frac{\text{SNR}_i}{1 + \text{INR}_i}\right) \\ &\quad - \log(1 + \text{INR}_{i+1} + \text{SNR}_{ip}) + 1 \\ &\leq 1 \end{aligned} \quad (147)$$

- $\gamma_i - g_i = 1$, because

$$\begin{aligned} \gamma_i - g_i &= \log(1 + \text{INR}_{i+1} + \text{SNR}_i) \\ &\quad - \log(1 + \text{INR}_{i+1} + \text{SNR}_i) + 1 \\ &= 1 \end{aligned} \quad (148)$$

- $\mu_i - e_{i-1} < 1$, because

$$\begin{aligned} \mu_i - e_{i-1} &= \log(1 + \text{INR}_i) \\ &\quad - \log(1 + \text{INR}_i + \text{SNR}_{i-1,p}) + 1 \\ &\leq 1 \end{aligned} \quad (149)$$

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